Exact Integral Formulation for Radiative Transfer in an Inhomogeneous Scattering Medium

Chih-Yang Wu*
National Cheng Kung University, Tainan, Taiwan, Republic of China

The exact integral formulation in terms of the moments of intensity is developed for radiative transfer in a three-dimensional, inhomogeneous medium with anisotropic scattering. Since the use of the present formulation in radiation computation requires the integration only over spatial variables, it reduces the labor of computation. Examples, including isotropic scattering and anisotropic scattering in a slab, and isotropic scattering in a two-dimensional, rectangular medium exposed to collimated radiation, are presented to show the effectiveness of the present formulation. The examples are solved by the Nyström method. Numerical results for the one-dimensional, isotropically scattering case show excellent agreement with available solutions reported in the literature. Graphic results of both one- and two-dimensional examples are presented to show the effects of the space-dependent albedo.

Nomenclature

A_i	= visible portion of the boundary exposed to incident radiation
a_n	= expansion coefficient in the phase function
B_j, C_k	= expansion coefficients in the space-dependent albedo
I	= intensity
I_b	= Planck function for emission
I_c	= intensity of collimated radiation
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	= unit vectors
M_{nm}, M^*_{nm}	= moments of intensity
M_{00}	= total intensity
M_{10}	= z component of the radiative flux
M^+_{10}	= half-range flux in positive z direction
M^-_{10}	= half-range flux in negative z direction
M_{11}	= x component of the radiative flux
M^*_{11}	= y component of the radiative flux
N_q	= number of Gaussian quadrature points
$\boldsymbol{n}_i$	= unit inward normal vector
P^m_n	= associated Legendre function
p	= phase function
$\boldsymbol{q}_r$	= radiative heat flux
R	= reflectance
$\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{r}''$	= position vectors (see Fig. 2)
$\boldsymbol{r}_i$	= location of the incident radiation
S	= source function
s	= coordinate along a ray (see Fig. 1)
s_0	= coefficient [see Eq. (31)]
T	= transmittance
V	= visible portion of the medium
x,y,z	= coordinates
β	= extinction coefficient
θ	= polar angle
θ_i	= polar angle of incident radiation
κ	= absorption coefficient
μ	= $\cos\theta$
μ_i	= $\cos\theta_i$
ν	= frequency
σ	= scattering coefficient
τ_a	= one-half of the optical size in x direction
τ_b	= one-half of the optical size in y direction
τ_c	= optical thickness
τ_x, τ_y, τ_z	= optical coordinates
φ	= azimuthal angle
φ_i	= azimuthal angle of incident radiation
Ω	= solid angle
$\boldsymbol{\Omega}$	= unit vector along the path of a ray
$\boldsymbol{\Omega}_i$	= unit vector along the direction of incident radiation
ω	= scattering albedo
ω_0	= coefficient [see Eq. (31)]

Subscripts

i	= incidence
x,y,z	= x,y,z components, respectively

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*Associate Professor, Department of Mechanical Engineering.

Introduction

In the last three decades, interest in the multidimensional aspects of radiative heat transfer in scattering media has increased greatly. This is because multidimensional scattering can be significant in the problem of energy transport in pulverized-fuel-fired furnaces, metalized propellant plumes, waste-heat extraction, and planetary atmosphere. A comprehensive literature survey of the radiative transfer investigations has been presented by Crosbie and Linsenbardt[1] and Viskanta and Mengüc.[2] A review of the literature survey reveals that most of the investigations have assumed the properties of the media to be constant to simplify analyses. However, the radiation properties of the media in practical applications depend strongly on location; thereby, in addition to scattering, the space dependence of the properties also plays an important role. Because consideration of variable properties adds to the complexity of the problem, only a limited number of investigations have been done in the area of inhomogeneous scattering media.

Some numerical results for one-dimensional, isotropic scattering in a medium with space-dependent properties have been presented recently. Garcia and Siewert[3] applied the F_N method to generate highly accurate solutions for isotropic scattering in inhomogeneous plane-parallel atmospheres. Özisik and coworkers used the Galerkin's method to investigate isotropic scattering in a slab[4] and in a sphere[5] with space-dependent albedo. El Wakil et al.[6] solved the equation of radiative transfer in an inhomogeneous, finite, plane-parallel medium by a bivariational technique.

A few formulation works for multidimensional scattering in inhomogeneous media have been presented in the literature. Mengüc and Viskanta[7] presented a differential approximation for three-dimensional scattering in inhomogeneous media and results for radiative transfer in a homogeneous medium. Lin[8] reported integral expressions for isotropic scattering and solved special cases for homogeneous media. Lin and Tsai[9] developed an integral formulation of anisotropic scattering in an arbitrary inhomogeneous medium in terms of the intensity and the source function, which are functions of location and direction.

The purpose of this analysis is twofold. First, we use the transformation of integration to develop integral formulation in terms of the moments of intensity for radiative transfer in a three-dimensional, absorbing, emitting, and anisotropically scattering medium with space-dependent properties. Second, computational examples for one- and two-dimensional media with space-dependent albedo are presented to illustrate the application of the present formulation.

The integral formulation in terms of the moments of intensity is pioneered by Hunt.[10] He used a complicated split-intensity technique to derive the integral formulation. Recently, Wu used a straightforward transformation method to derive the integral formulation for linearly anisotropic scattering in a three-dimensional medium with Fresnel boundaries[11] and for anisotropic scattering of arbitrary order.[12] In this work, we follow a procedure similar to that of Wu[11,12] to derive the integral formulation involving the space-dependent properties. Because the moments of intensity are independent of direction, the present formulation reduces the labor of computation, especially for anisotropic scattering cases.

As illustrations, we present the integral formulation for anisotropic scattering in a three-dimensional, rectangular medium exposed to collimated radiation and then solve the one- and two-dimensional versions of the formulation. Gaussian quadrature formula is used to transform the integral equations into a system of algebraic equations. Three examples are considered. First, numerical solutions of one-dimensional, isotropic scattering in a slab with space-dependent albedo, for which accurate numerical results are available, are presented to illustrate the validity and accuracy of the present method. Second, linearly anisotropic scattering in a slab is studied. Third, isotropic scattering in a two-dimensional rectangular medium is considered. Graphic results are presented to show the effects of space-dependent albedo for various optical sizes. No comparisons are made for the last two examples, since to our knowledge, there are no published results for those cases.

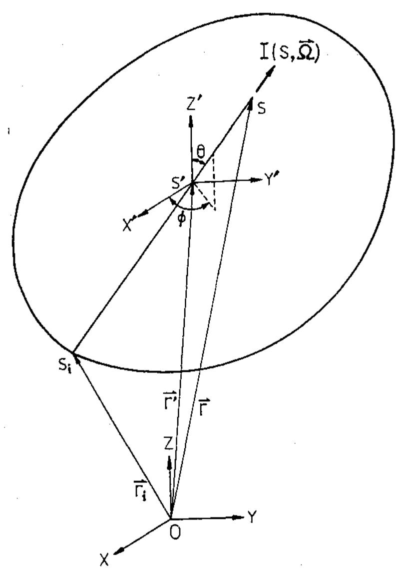

Fig. 1 Radiation intensity along a path in a general three-dimensional geometry.

General Formulation

The geometry and coordinates for the present formulation are shown in Fig. 1. The radiative transfer equation for the radiative intensity I along an arbitrary direction in an emitting, absorbing, and scattering medium in local thermodynamic equilibrium can be expressed as

$$\frac{\mathrm{d}I(s,\mathbf{\Omega})}{\mathrm{d}s} + \beta(s)I(s,\mathbf{\Omega}) = \beta(s)S(s,\mathbf{\Omega}) \tag{1}$$

S is the source function defined by

$$S(s,\mathbf{\Omega}) = [1-\omega(s)]I_b(s) + \frac{\omega(s)}{4\pi}\int_{4\pi} p(\mathbf{\Omega}',\mathbf{\Omega})I(s,\mathbf{\Omega}')\,\mathrm{d}\Omega' \tag{2}$$

Here, ω is the albedo defined by σ/β, I_b is the Planck function for emission, $\mathrm{d}\Omega'$ is an element of solid angle, and $p(\mu',\varphi',\mu,\varphi)$ is the phase function approximated by a series of spherical harmonics[13]

$$p(\mu',\varphi',\mu,\varphi) = \sum_{m=0}^{N}\sum_{n=m}^{N}(2-\delta_{0m})a_{nm}P_n^m(\mu) \times P_n^m(\mu')\cos[m(\varphi-\varphi')] \tag{3a}$$

where

$$a_{nm} = a_n\frac{(n-m)!}{(n+m)!} \quad \text{with } a_0 = 1;$$

$$m \le n \le N; \quad 0 \le m \le N \tag{3b}$$

$$\delta_{0m} = \begin{cases}1 & \text{for } m=0\\ 0 & \text{otherwise}\end{cases} \tag{3c}$$

$$\mu = \cos(\theta) \tag{3d}$$

The series is truncated after the Nth term for Nth-degree, anisotropic scattering. The subscript ν, which denotes the spectrally dependent properties of the medium and the boundaries, is omitted to simplify the mathematical expressions.

Substitution of Eq. (3) into Eq. (2) yields

$$S(s,\mu,\varphi) = [1-\omega(s)]I_b(s) + \frac{\omega(s)}{4\pi}\sum_{m=0}^{N}\sum_{n=m}^{N}(2-\delta_{0m})a_{nm}P_n^m(\mu) \times [\cos(m\varphi)M_{nm}(s) + \sin(m\varphi)M_{nm}^*(s)] \tag{4}$$

where the moments of intensity are defined by

$$M_{nm}(s) = \int_{\varphi'=0}^{2\pi}\int_{\mu'=-1}^{1} I(s,\mu',\varphi')P_n^m(\mu')\cos(m\phi')\,\mathrm{d}\mu'\,\mathrm{d}\varphi'$$

$$m \le n \le N; \qquad 0 \le m \le N \tag{5}$$

and

$$M_{nm}^*(s) = \int_{\varphi'=0}^{2\pi}\int_{\mu'=-1}^{1} I(s,\mu',\varphi')P_n^m(\mu')\sin(m\varphi')\,\mathrm{d}\mu'\,\mathrm{d}\varphi'$$

$$m \le n \le N; \qquad 0 \le m \le N \tag{6}$$

For a given coordinate system, Eq. (4) may be expressed as

$$S(\boldsymbol{r},\mathbf{\Omega}) = [1-\omega(\boldsymbol{r})]I_b(\boldsymbol{r}) + \frac{\omega(\boldsymbol{r})}{4\pi}\sum_{m=0}^{N}\sum_{n=m}^{N}(2-\delta_{0m})a_{nm}P_n^m(\mu) \times [\cos(m\varphi)M_{nm}(\boldsymbol{r}) + \sin(m\varphi)M_{nm}^*(\boldsymbol{r})] \tag{7}$$

where $\boldsymbol{r}$ denotes the location of the point studied,

$$\mathbf{\Omega} = (\boldsymbol{r}''-\boldsymbol{r})/|\boldsymbol{r}''-\boldsymbol{r}| \tag{8}$$

denotes the direction of the ray from $\boldsymbol{r}$ to $\boldsymbol{r}''$ (see Fig. 2), and the trigonometric quantities $\sin m\varphi$, $\cos m\varphi$ and μ can be expressed as the functions of the directional cosines

$$(1-\mu^2)^{1/2}\cos(\varphi)=\boldsymbol{\Omega}\cdot\boldsymbol{i} \tag{9}$$

$$(1-\mu^2)^{1/2}\sin(\varphi)=\boldsymbol{\Omega}\cdot\boldsymbol{j} \tag{10}$$

$$\mu=\boldsymbol{\Omega}\cdot\boldsymbol{k} \tag{11}$$

The integration of Eq. (1) along the path of a ray yields a formal solution of intensity. Substituting the formal solution of intensity into Eqs. (5) and (6) using Eq. (4) and transforming the integrals in the resulting equations into surface integrals and volume integrals,[12] we can obtain the integral equations for the moments of intensity as follows:

$$\begin{aligned}M_{nm}(\boldsymbol{r})=&\int_A I(\boldsymbol{r}_i,\boldsymbol{\Omega}_i)P_n^m(\mu_i)\cos(m\varphi_i)\\&\times\exp\left\{-\int_0^1\beta[\boldsymbol{r}+\xi(\boldsymbol{r}_i-\boldsymbol{r})]|\boldsymbol{r}_i-\boldsymbol{r}|\,\mathrm{d}\xi\right\}\frac{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{n}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\mathrm{d}A_i\\&+\int_V\beta[\boldsymbol{r}']P_n^m(\mu')\cos(m\varphi')\Big\{[1-\omega(\boldsymbol{r}')]I_b(\boldsymbol{r}')\\&+\frac{\omega(\boldsymbol{r}')}{4\pi}\sum_{k=0}^{N}\sum_{j=k}^{N}(2-\delta_{0j})a_{kj}P_k^j(\mu')\\&\times[\cos(j\varphi')M_{kj}(\boldsymbol{r}')+\sin(j\varphi')M_{kj}^*(\boldsymbol{r}')]\Big\}\\&\times\exp\left\{-\int_0^1\beta[\boldsymbol{r}+\xi'(\boldsymbol{r}'-\boldsymbol{r})]|\boldsymbol{r}'-\boldsymbol{r}|\,\mathrm{d}\xi'\right\}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\mathrm{d}V'\\&m\le n\le N;\qquad 0\le m\le N\end{aligned} \tag{12}$$

$$\begin{aligned}M_{nm}^*(\boldsymbol{r})=&\int_A I(\boldsymbol{r}_i,\boldsymbol{\Omega}_i)P_n^m(\mu_i)\sin(m\varphi_i)\\&\times\exp\left\{-\int_0^1\beta[\boldsymbol{r}+\xi(\boldsymbol{r}_i-\boldsymbol{r})]|\boldsymbol{r}_i-\boldsymbol{r}|\,\mathrm{d}\xi\right\}\frac{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{n}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\mathrm{d}A_i\\&+\int_V\beta[\boldsymbol{r}']P_n^m(\mu')\sin(m\varphi')\Big\{[1-\omega(\boldsymbol{r}')]I_b(\boldsymbol{r}')\\&+\frac{\omega(\boldsymbol{r}')}{4\pi}\sum_{k=0}^{N}\sum_{j=k}^{N}(2-\delta_{0j})a_{kj}P_k^j(\mu')\\&\times[\cos(k\varphi')M_{kj}(\boldsymbol{r}')+\sin(j\varphi')M_{kj}^*(\boldsymbol{r}')]\Big\}\\&\times\exp\left\{-\int_0^1\beta[\boldsymbol{r}+\xi'(\boldsymbol{r}'-\boldsymbol{r})]|\boldsymbol{r}'-\boldsymbol{r}|\,\mathrm{d}\xi'\right\}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\mathrm{d}V'\\&m\le n\le N;\qquad 0\le m\le N\end{aligned} \tag{13}$$

where $\mathrm{d}A_i$ denotes an infinitesimal area about the incident location $\boldsymbol{r}_i$ with unit inward normal vector $\boldsymbol{n}_i$, $\mathrm{d}V'$ denotes an infinitesimal volume about the point $\boldsymbol{r}'$,

$$\boldsymbol{\Omega}_i=(\boldsymbol{r}-\boldsymbol{r}_i)/|\boldsymbol{r}-\boldsymbol{r}_i| \tag{14}$$

denotes the direction of the incident radiation on the boundary,

$$\boldsymbol{\Omega}'=(\boldsymbol{r}-\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'| \tag{15}$$

denotes the direction of the ray from $\boldsymbol{r}'$ to $\boldsymbol{r}$ (see Fig. 2), and the trigonometric quantities $\sin m\varphi_i$, $\cos m\varphi_i$, $\sin m\varphi'$, $\cos m\varphi'$, μ_i, and μ' can be expressed as the functions of the directional cosines

$$(1-\mu_i^2)^{1/2}\cos(\varphi_i)=\boldsymbol{\Omega}_i\cdot\boldsymbol{i} \tag{16}$$

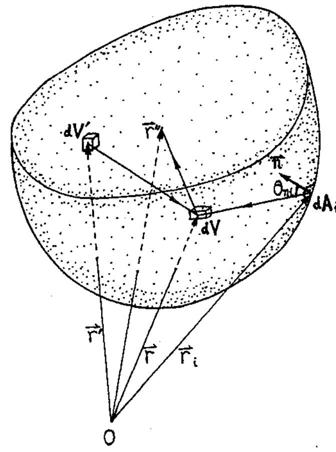

Fig. 2 Position and direction vectors for the derivation of the integral formulation.

$$(1-\mu_i^2)^{1/2}\sin(\varphi_i)=\boldsymbol{\Omega}_i\cdot\boldsymbol{j} \tag{17}$$

$$\mu_i=\boldsymbol{\Omega}_i\cdot\boldsymbol{k} \tag{18}$$

$$(1-\mu'^2)^{1/2}\cos(\varphi')=\boldsymbol{\Omega}'\cdot\boldsymbol{i} \tag{19}$$

$$(1-\mu'^2)^{1/2}\sin(\varphi')=\boldsymbol{\Omega}'\cdot\boldsymbol{j} \tag{20}$$

$$\mu'=\boldsymbol{\Omega}'\cdot\boldsymbol{k} \tag{21}$$

If the boundary is opaque, A_i and V in Eqs. (12) and (13) are, respectively, the visible portions of the boundary and the medium. For a concave medium enclosed by transparent boundaries, the effects of re-entrant have yet to be accounted for. Physically, the first four moments of intensity are of importance. M_{00} is the total intensity, whereas the radiative flux can be expressed as

$$\boldsymbol{q}_r(\boldsymbol{r})=M_{11}(\boldsymbol{r})\boldsymbol{i}+M_{11}^*(\boldsymbol{r})\boldsymbol{j}+M_{10}(\boldsymbol{r})\boldsymbol{k} \tag{22}$$

Equations (12) and (13) form a complete description of the present problem, provided that the incident radiation is known. Substitution of Eqs. (14) and (15) into Eqs. (16–21) shows that all of the angular quantities can be replaced by spatial quantities. Thus, the moments of intensity are the dependent variables of the present problem, and they are the functions of spatial variables.

In thermal radiation computation, once we obtain the solutions of Eqs. (12) and (13), we can use Eqs. (7) and (22) to find the source function and the radiative flux, respectively. Since the use of the present formulation requires the integration only over spatial variables, it reduces the labor of computation.

If we consider isotropic scattering, following a procedure similar to that of Wu,[12] we can reduce Eqs. (12) and (13) to the integral equation of the source function

$$\begin{aligned}S(\boldsymbol{r})=&[1-\omega(\boldsymbol{r})]I_b(\boldsymbol{r})+\frac{\omega(\boldsymbol{r})}{4\pi}\int_A I(\boldsymbol{r}_i,\boldsymbol{\Omega}_i)\\&\times\exp\left\{-\int_0^1\beta[\boldsymbol{r}+\xi(\boldsymbol{r}_i-\boldsymbol{r})]|\boldsymbol{r}_i-\boldsymbol{r}|\,\mathrm{d}\xi\right\}\frac{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{n}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\mathrm{d}A_i\\&+\frac{\omega(\boldsymbol{r})}{4\pi}\int_V\beta[\boldsymbol{r}']S(\boldsymbol{r}')\\&\times\exp\left\{-\int_0^1\beta[\boldsymbol{r}+\xi'(\boldsymbol{r}'-\boldsymbol{r})]|\boldsymbol{r}'-\boldsymbol{r}|\,\mathrm{d}\xi'\right\}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\mathrm{d}V'\end{aligned} \tag{23}$$

Equation (23) corresponds to Eq. (1) of Lin's work.[8] This result shows that the special case of the present formulation is consistent with Lin's formulation for isotropic scattering.[8]

Computational Examples and Discussion

To illustrate the effectiveness of the present formulation, we apply the formulation to scattering in rectangular media. One- and two-dimensional versions of the formulation are solved by an exact numerical method. For the purpose of comparison with available solutions, the extinction coefficient is assumed to be constant, and the spatial variation of albedo is considered. Such special cases have been investigated by several authors.[3-6]

Here, we begin at the formulation for radiative transfer in a three-dimensional absorbing and linearly anisotropically scattering medium exposed to collimated radiation of intensity I_c. To simplify the formulation, only the incident radiation at $\tau_z = 0$ in the direction $\theta = \theta_i$ and $\varphi = \varphi_i$ is considered. After defining the optical variables $\tau_x = \beta x$, $\tau_y = \beta y$, and $\tau_z = \beta z$, we choose the optical coordinate system such that the medium is bounded at $\tau_x = \pm\tau_a$, $\tau_y = \pm\tau_b$, and $\tau_z = 0, \tau_c$. Then, Eqs. (14) and (15) become

$$\boldsymbol{\Omega}_i = \sin(\theta_i)\cos(\phi_i)\boldsymbol{i} + \sin(\theta_i)\sin(\phi_i)\boldsymbol{j} + \mu_i\boldsymbol{k} \tag{24}$$

$$\boldsymbol{\Omega}' = [(\tau_x - \tau'_x)\boldsymbol{i} + (\tau_y - \tau'_y)\boldsymbol{j} + (\tau_z - \tau'_z)\boldsymbol{k}]/\tau_r \tag{25}$$

with

$$\tau_r = [(\tau_x - \tau'_x)^2 + (\tau_y - \tau'_y)^2 + (\tau_z - \tau'_z)^2]^{1/2} \tag{26}$$

Using Eqs. (12) and (13), we obtain the integral equations for the present problem

$$\begin{aligned} M_{00}(\tau_x,\tau_y,\tau_z) &= I_c[\tau_x - \tau_z\tan(\theta_i)\cos(\varphi_i), \\ &\quad \tau_y - \tau_z\tan(\theta_i)\sin(\varphi_i)]\exp(-\tau_z/\mu_i) \\ &\quad + \int_0^{\tau_c}\int_{-\tau_b}^{\tau_b}\int_{-\tau_a}^{\tau_a} \frac{\exp(-\tau_r)}{\tau_r^2}\frac{\omega(\tau'_x,\tau'_y,\tau'_z)}{4\pi}\Big\{M_{00}(\tau'_x,\tau'_y,\tau'_z) \\ &\quad + \frac{a_1}{\tau_r}[(\tau_z - \tau'_z)M_{10}(\tau'_x,\tau'_y,\tau'_z) + (\tau_x - \tau'_x)M_{11}(\tau'_x,\tau'_y,\tau'_z) \\ &\quad + (\tau_y - \tau'_y)M^*_{11}(\tau'_x,\tau'_y,\tau'_z)]\Big\}\,d\tau'_x\,d\tau'_y\,d\tau'_z \end{aligned} \tag{27}$$

$$\begin{aligned} M_{10}(\tau_x,\tau_y,\tau_z) &= I_c[\tau_x - \tau_z\tan(\theta_i)\cos(\varphi_i), \\ &\quad \tau_y - \tau_z\tan(\theta_i)\sin(\varphi_i)]\mu_i\exp(-\tau_z/\mu_i) \\ &\quad + \int_0^{\tau_c}\int_{-\tau_b}^{\tau_b}\int_{-\tau_a}^{\tau_a} \frac{\exp(-\tau_r)}{\tau_r^3}(\tau_z - \tau'_z) \\ &\quad \times\frac{\omega(\tau'_x,\tau'_y,\tau'_z)}{4\pi}\Big\{M_{00}(\tau'_x,\tau'_y,\tau'_z) \\ &\quad + \frac{a_1}{\tau_r}[(\tau_z - \tau'_z)M_{10}(\tau'_x,\tau'_y,\tau'_z) + (\tau_x - \tau'_x)M_{11}(\tau'_x,\tau'_y,\tau'_z) \\ &\quad + (\tau_y - \tau'_y)M^*_{11}(\tau'_x,\tau'_y,\tau'_z)]\Big\}\,d\tau'_x\,d\tau'_y\,d\tau'_z \end{aligned} \tag{28}$$

$$\begin{aligned} M_{11}(\tau_x,\tau_y,\tau_z) &= I_c[\tau_x - \tau_z\tan(\theta_i)\cos(\varphi_i), \\ &\quad \tau_y - \tau_z\tan(\theta_i)\sin(\varphi_i)]\sin(\theta_i)\cos(\varphi_i)\exp(-\tau_z/\mu_i) \\ &\quad + \int_0^{\tau_c}\int_{-\tau_b}^{\tau_b}\int_{-\tau_a}^{\tau_a} \frac{\exp(-\tau_r)}{\tau_r^3}(\tau_x - \tau'_x) \\ &\quad \times\frac{\omega(\tau'_x,\tau'_y,\tau'_z)}{4\pi}\Big\{M_{00}(\tau'_x,\tau'_y,\tau'_z) + \frac{a_1}{\tau_r}[(\tau_z - \tau'_z)M_{10}(\tau'_x,\tau'_y,\tau'_z) \\ &\quad + (\tau_x - \tau'_x)M_{11}(\tau'_x,\tau'_y,\tau'_z) \\ &\quad + (\tau_y - \tau'_y)M^*_{11}(\tau'_x,\tau'_y,\tau'_z)]\Big\}\,d\tau'_x\,d\tau'_y\,d\tau'_z \end{aligned} \tag{29}$$

$$\begin{aligned} M^*_{11}(\tau_x,\tau_y,\tau_z) &= I_c[\tau_x - \tau_z\tan(\theta_i)\cos(\varphi_i), \\ &\quad \tau_y - \tau_z\tan(\theta_i)\sin(\varphi_i)]\sin(\theta_i)\sin(\varphi_i)\exp(-\tau_z/\mu_i) \\ &\quad + \int_0^{\tau_c}\int_{-\tau_b}^{\tau_b}\int_{-\tau_a}^{\tau_a} \frac{\exp(-\tau_r)}{\tau_r^3}(\tau_y - \tau'_y) \\ &\quad \times\frac{\omega(\tau'_x,\tau'_y,\tau'_z)}{4\pi}\Big\{M_{00}(\tau'_x,\tau'_y,\tau'_z) \\ &\quad + \frac{a_1}{\tau_r}[(\tau_z - \tau'_z)M_{10}(\tau'_x,\tau'_y,\tau'_z) + (\tau_x - \tau'_x)M_{11}(\tau'_x,\tau'_y,\tau'_z) \\ &\quad + (\tau_y - \tau'_y)M^*_{11}(\tau'_x,\tau'_y,\tau'_z)]\Big\}\,d\tau'_x\,d\tau'_y\,d\tau'_z \end{aligned} \tag{30}$$

When the incident radiation and the albedo are independent of τ_x, and the medium is unbounded in the $\pm\tau_x$ direction, Eqs. (27–30) can be simplified to their two-dimensional form. Similarly, when I_c and ω are independent of τ_x and τ_y and the medium is unbounded in $\pm\tau_x$ and $\pm\tau_y$ directions, we have the one-dimensional version of the present formulation.

Equations (27–30) or their one- or two-dimensional versions are the Fredholm integral equations of the second kind. Highly accurate numerical solutions of the equations can be obtained by using a number of methods, such as the finite element methods,[8,9,14] the Galerkin's method,[15] and the Nyström method[16] with the removal of singularity.[17,18] The accuracy of the Nyström method with the removal of singularity has been shown to be high for radiative transfer in homogeneous media.[17,18] Thus, the method is adopted in the following examples.

The first example considered is radiative transfer in an isotropically scattering slab with an exponential variation of albedo

$$\omega(\tau_z) = \omega_0\exp(-\tau_z/s_0) \tag{31}$$

where $0 < \omega_0 \le 1$ and $s_0 > 0$. Table 1 shows the reflectance and the transmittance of a slab exposed to unit collimated radiation with $\mu_i = 0.9$. The reflectance is simply $M^-_{10}(0)$, and the transmittance $M^+_{10}(\tau_c)$. Excellent agreement of the results by the present method with those by the F_N method[3] shows the validity of the present formulation. It is found that the number of quadrature points required to generate accurate solutions increases with the optical thickness of the slab. This conclusion has also been found in the problems for homogeneous media.[11] Moreover, the present method generates a numerically exact solution, provided that the number of quadrature points is large enough.

In the second example, linear anisotropic scattering in a slab with the albedo

$$\omega(\tau_z) = \sum_{k=0}^{K} C_k\tau_z^k \tag{32}$$

is studied. Here, the constants C_k shall satisfy the requirement that $0 \le \omega(\tau_z) \le 1$. The R-τ_c curves and the T-τ_c curves of a slab with normal incidence and several different spatial variations of albedo, which have the same average value, are shown in Figs. 3 and 4, respectively. The R-τ_c curves in Fig. 3 show a strong dependence of reflectance on the spatial distributions of albedo, and the T-τ_c curves in Fig. 4 show a weak dependence of transmittance on the distributions of albedo. The same trends also appear in isotropically scattering cases.[4] The influence of the coefficient of anisotropic scattering on reflectance is far stronger than that on transmittance for all albedos considered. The same trend continues for all optical thickness as shown in Figs. 3 and 4.

Finally, isotropic scattering in a two-dimensional, rectangular, inhomogeneous medium exposed to unit-normal incidence at $\tau_z = 0$ is considered. The spatial distribution of albedo is

Table 1 The reflectance and the transmittance for a slab exposed to collimated incidence ($\mu_i = 0.9$)

		Reflectance		Transmittance	
ω_0	s_0	This work	F_N method[3]	This work	F_N method[3]
		a) $\tau_c = 0.1$, $N_q = 12$			
0.7	1	0.0330938	0.0330938	0.9275856	0.927586
	10	0.0346827	0.0346827	0.9293223	0.929322
	100	0.0348478	0.0348478	0.9295030	0.929503
	1000	0.0348644	0.0348644	0.9295212	0.929521
	∞	0.0348662	0.0348662	0.9295232	0.929523
1.0	1	0.0498765	0.0498765	0.9442008	0.944201
	10	0.0524177	0.0524177	0.9469669	0.944967
	100	0.0526825	0.0526825	0.9472557	0.947256
	1000	0.0527091	0.0527091	0.9472847	0.947258
	∞	0.0527121	0.0527121	0.9472879	0.947288
		b) $\tau_c = 1.0$, $N_q = 18$			
0.7	1	0.1150063	0.115005	0.3944458	0.394446
	10	0.1684180	0.168417	0.4575467	0.457548
	100	0.1770222	0.177022	0.4689095	0.468911
	1000	0.1779373	0.177937	0.4701340	0.470135
	∞	0.1780396	0.178039	0.4702711	0.470272
1.0	1	0.1971058	0.197101	0.4421925	0.442193
	10	0.3340147	0.334012	0.5960817	0.596084
	100	0.3616798	0.361677	0.6306002	0.630603
	1000	0.3647454	0.364743	0.6344744	0.634477
	∞	0.3650898	0.365087	0.6349102	0.634913
		c) $\tau_c = 10.0$, $N_q = 96$			
0.7	1	0.1188541	0.118853	0.0000170	0.0000170
	10	0.1969864	0.196985	0.0000516	0.0000516
	100	0.2167876	0.216786	0.0002105	0.0002105
	1000	0.2192405	0.219239	0.0002801	0.0002802
	∞	0.2195206	0.219519	0.0002901	0.0002902
1.0	1	0.2051770	0.205174	0.0000185	0.0000185
	10	0.4662703	0.466262	0.0001911	0.0001911
	100	0.6984490	0.698436	0.0187668	0.0187701
	1000	0.8284139	0.820402	0.1028043	0.102816
	∞	0.8619751	0.861963	0.1380242	0.138037

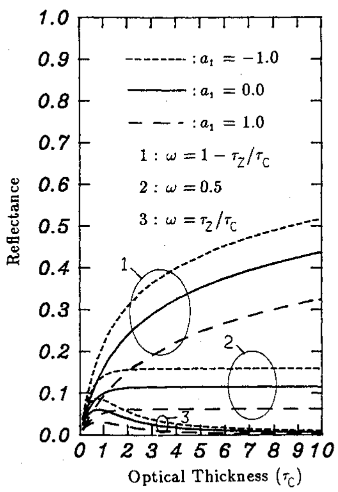

Fig. 3 Effects of anisotropic scattering and spatial variation of albedo on reflectance of a slab exposed to normal incidence.

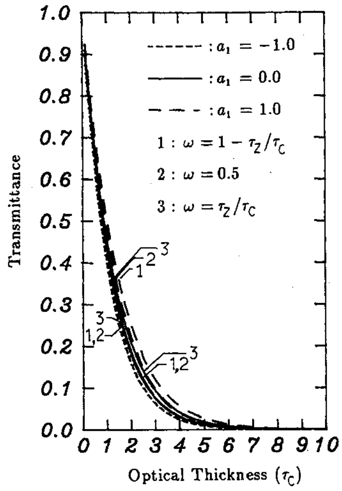

Fig. 4 Effects of anisotropic scattering and spatial variation of albedo on transmittance of a slab exposed to normal incidence.

expressed as

$$\omega(\tau_y,\tau_z) = \sum_{j=0}^{J} \sum_{k=0}^{K} B_j C_k \tau_y^j \tau_z^k \tag{33}$$

where B_j and C_k are constants and $0 \le \omega(\tau_y,\tau_z) \le 1$. This problem is described by the two-dimensional version of Eqs. (27–30). The graphic results shown in Figs. 5–8 are obtained by the application of 18 × 22 Gaussian quadrature points. Figure 5 shows that the dependence of $M^-_{\bar{1}0}(\tau_y,0)$ (the "reflectance") on the spatial distribution of albedo is stronger than the dependence of $M^+_{\bar{1}0}(\tau_y,\tau_c)$ (the "transmittance") on the distribution of albedo. This trend is the same as that found in the one-dimensional results shown in Figs. 3 and 4. The very strong dependence of $M^+_{\bar{1}1}(\tau_b,\tau_z)$ on the variation of albedo in the τ_z direction is shown in Fig. 6. Figure 7 shows that the dependence of the reflectance on the optical size is weaker than the dependence of the transmittance on the optical size

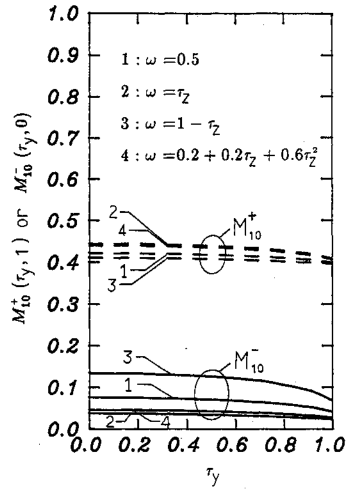

Fig. 5 Effects of spatial variation of albedo on the radiative flux leaving the top and bottom surface of a rectangular medium ($\tau_b = 0.5, \tau_c = 1$).

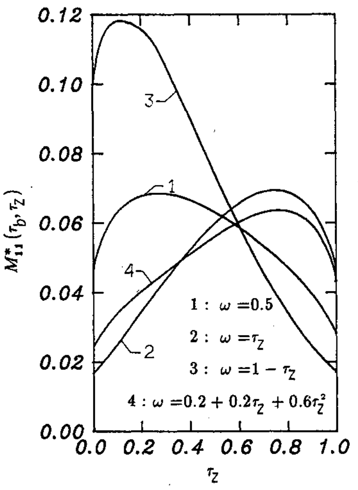

Fig. 6 Effects of spatial variation of albedo on the radiative flux leaving the side surface of a rectangular medium ($\tau_b = 0.5, \tau_c = 1$).

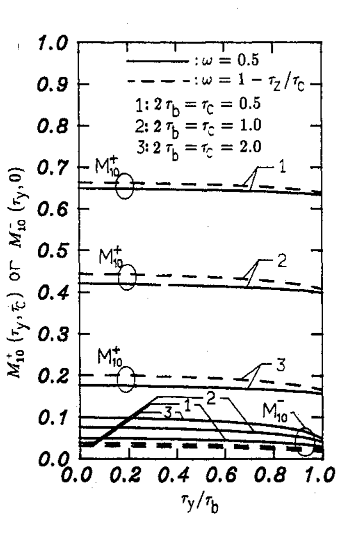

Fig. 7 Effects of optical size on the radiative flux leaving the top and bottom surface of a rectangular medium.

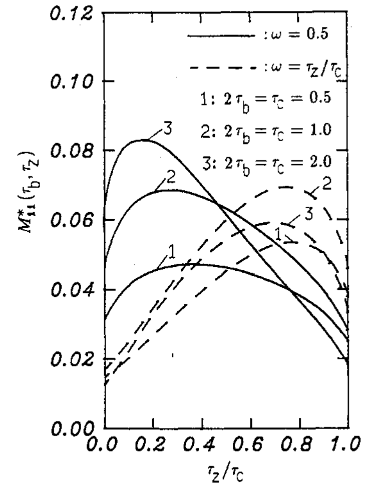

Fig. 8 Effects of optical size on the radiative flux leaving the side surface of a rectangular medium.

for various albedos. In Fig. 8, the curves for radiative transfer in homogeneous media with $\omega = 0.5$ shows that the variation of $M_{11}^*(\tau_b,\tau_z)$ in the τ_z direction increases with the increase of the optical size, but the curves for radiative transfer in inhomogeneous media with $\omega = \tau_z/\tau_c$ does not show the same trend. Because 0.5 is the average value of τ_z/τ_c, the qualitative difference shown in Fig. 8 means that an albedo with strong spatial variation cannot be approximated by its average value.

Concluding Remarks

The exact integral formulation in terms of the moments of intensity is developed for radiative transfer in a three-dimensional, inhomogeneous medium exposed to given incident radiation. Results of the examples considered show that the present formulation is valid at least for one- and two-dimensional cases. To extend the present formulation to a medium with specular boundaries, where the inward intensity is unknown, we can apply the image technique.[11] The exact integral equations in terms of the moments of intensity should be developed for the generalized case.

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